Warsaw University of Technology
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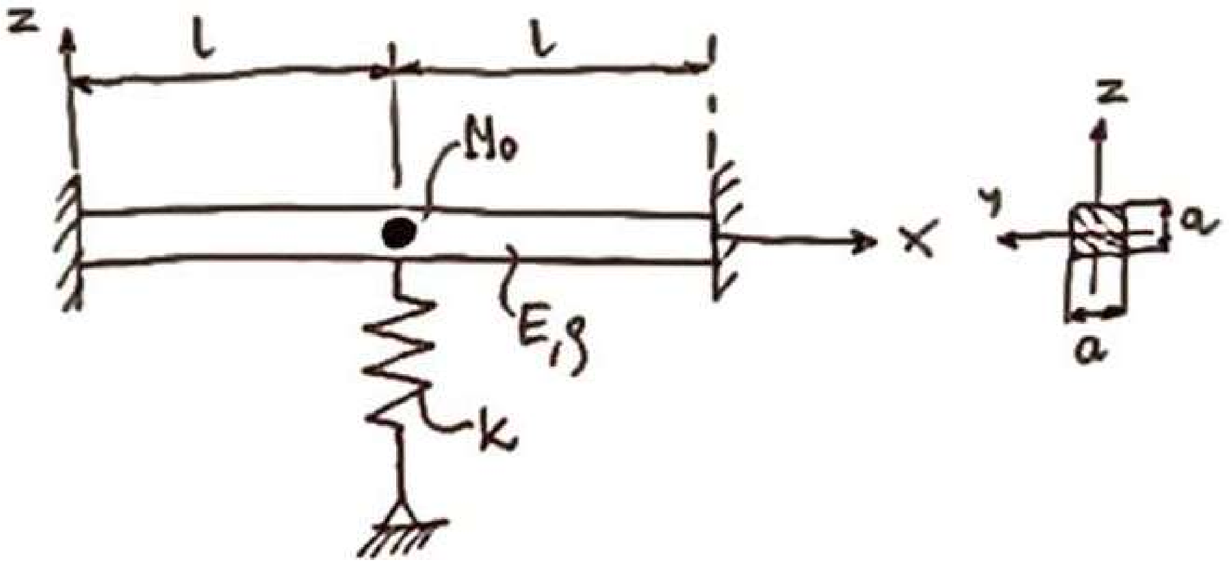
Finite element method 2 (FEM 2)

Example of modal analysis

11.2021

Example. Build a finite element model and write the set of equations for modal analysis of a 2D structure consisting of a beam, spring and mass M_0. The beam is represented by 2 finite elements.

a) Find natural frequencies (f_1, f_2) and corresponding eigenvectors. Draw the mode shapes.
b) Calculate the minimum value of spring stiffness, so that the mode shape without the translation of mass M_0 becomes the first vibration mode.



$E = 2\cdot 10^5$ MPa

$L = 1000$ mm

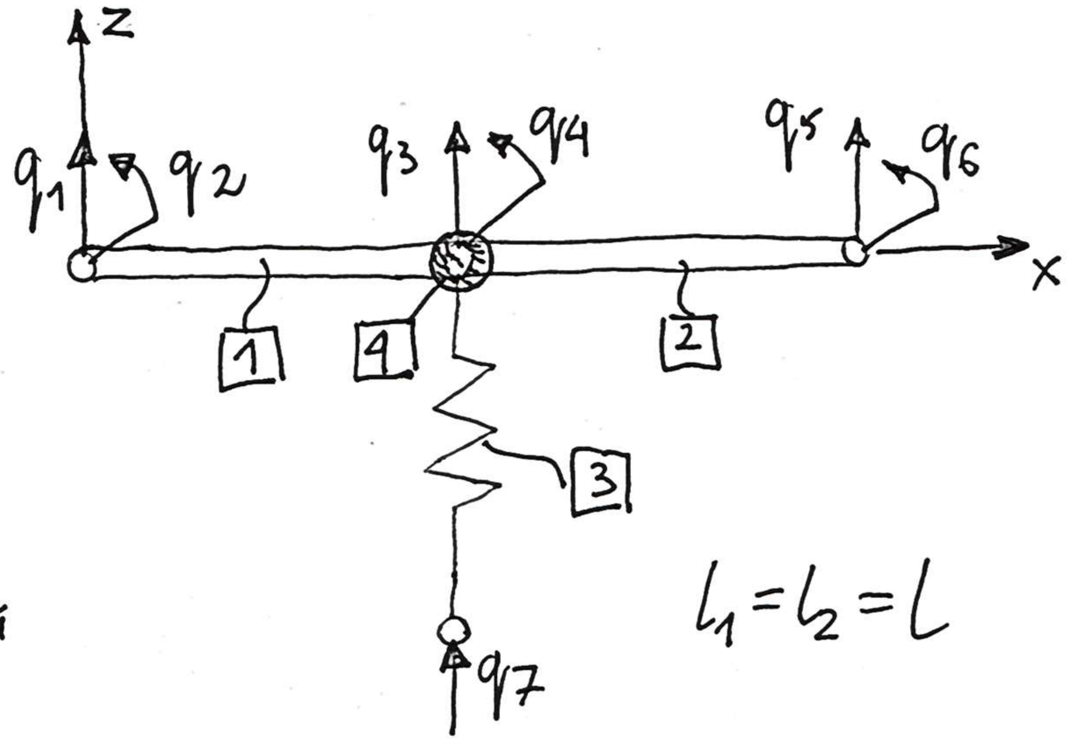
$a = 40$ mm

$\rho = 7800 \frac{kg}{m^3}$

$M_0 = 20$ kg

$k = 5000 \frac{N}{mm}$

$A = a^2$, $J_y = \frac{a^4}{12}$



$l_1 = l_2 = L$

GLOBAL VECTOR OF NODAL PARAMETERS :

$$\underset{1\times 7}{\lfloor q \rfloor} = \lfloor q_1, q_2, q_3, q_4, q_5, q_6, q_7 \rfloor$$

LOCAL STIFFNESS MATRICES:

$$\underset{4\times 4}{[k]_1} = \underset{4\times 4}{[k]_2} = \frac{2EJ_y}{l^3} \cdot \begin{bmatrix} 6 & 3l & -6 & 3l \\ 3l & 2l^2 & -3l & l^2 \\ -6 & -3l & 6 & -3l \\ 3l & l^2 & -3l & 2l^2 \end{bmatrix} = \frac{Ea^4}{6l^3} \cdot \begin{bmatrix} 6 & 3l & -6 & 3l \\ 3l & 2l^2 & -3l & l^2 \\ -6 & -3l & 6 & -3l \\ 3l & l^2 & -3l & 2l^2 \end{bmatrix}$$

$$\underset{2\times 2}{[k]_3} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \quad , \quad \underset{(1\times 1)}{[k]_4} = \underset{(1\times 1)}{[0]}$$

LOCAL MASS MATRICES:

$$\underset{4\times 4}{[m]_1} = \underset{4\times 4}{[m]_2} = \frac{\rho a^2 l}{420} \cdot \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l \\ 54 & 13l & 156 & -22l \\ -13l & -3l & -22l & 4l^2 \end{bmatrix}$$

$$\underset{2\times 2}{[m]_3} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad \underset{(1\times 1)}{[m]_4} = \underset{(1\times 1)}{[M_0]}$$

EXTENDED STIFFNESS MATRICES.

$$\underset{7\times7}{[k]_1^*} = \begin{bmatrix} \underset{4\times4}{[k]_1} & \begin{matrix} 0&0&0\\0&0&0\\0&0&0\\0&0&0 \end{matrix} \\ \begin{matrix} 0&0&0&0\\0&0&0&0\\0&0&0&0 \end{matrix} & \begin{matrix} 0&0&0\\0&0&0\\0&0&0 \end{matrix} \end{bmatrix} \; ; \; \underset{7\times7}{[k]_2^*} = \begin{bmatrix} \begin{matrix} 0&0\\0&0 \end{matrix} & \begin{matrix} 0&0&0&0\\0&0&0&0 \end{matrix} & \begin{matrix} 0\\0 \end{matrix} \\ \begin{matrix} 0&0\\0&0\\0&0\\0&0 \end{matrix} & \underset{4\times4}{[k]_2} & \begin{matrix} 0\\0\\0\\0 \end{matrix} \\ \begin{matrix} 0&0 \end{matrix} & \begin{matrix} 0&0&0&0 \end{matrix} & 0 \end{bmatrix}$$

$$\underset{7\times7}{[k]_3^*} = \begin{bmatrix} 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&k&0&0&0&-k \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&-k&0&0&0&k \end{bmatrix} \; ; \; \underset{7\times7}{[k]_4^*} = \underset{7\times7}{[0]}$$

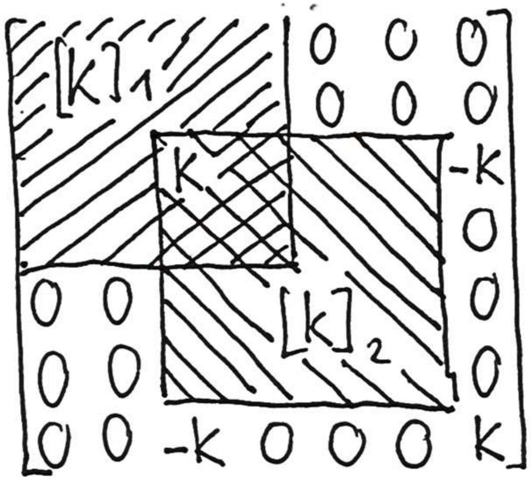
EXTENDED MASS MATRICES

$$\underset{7\times 7}{[m]_1^*} = \begin{bmatrix} \underset{4\times 4}{[m]_1} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \end{bmatrix} \; ; \qquad \underset{7\times 7}{[m]_2^*} = \begin{bmatrix} \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \underset{4\times 4}{[m]_2} & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 & 0 \end{matrix} & 0 \end{bmatrix}$$

$$\underset{7\times 7}{[m]_3^*} = \underset{7\times 7}{[0]} \; , \qquad \underset{7\times 7}{[m]_4^*} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & M_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

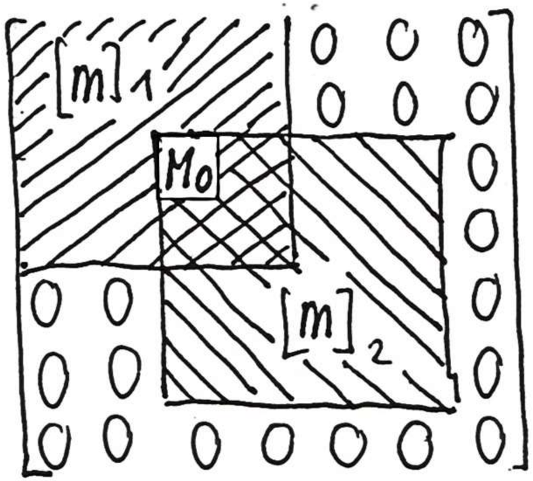
GLOBAL STIFFNESS MATRIX

$$\underset{7\times 7}{[K]} =$$



GLOBAL MASS MATRIX

$$\underset{7\times 7}{[M]} =$$



BOUNDARY CONDITIONS

$$q_1 = 0, \; q_2 = 0, \; q_5 = 0, \; q_6 = 0, \; q_7 = 0$$

SET OF EQUATIONS :

$$\left(\begin{bmatrix} K_{33\_1} + K_{11\_2} + k & K_{34\_1} + K_{12\_2} \\ K_{43\_1} + K_{21\_2} & K_{44\_1} + K_{22\_2} \end{bmatrix} - \omega^2 \begin{bmatrix} m_{33\_1} + m_{11\_2} + M_0 & m_{34\_1} + m_{12\_2} \\ m_{43\_1} + m_{21\_2} & m_{44\_1} + m_{22\_2} \end{bmatrix} \right) \cdot \begin{Bmatrix} q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$k_{33-1} + k_{11-2} + k = \frac{6Ea^4}{6l^3} + \frac{6Ea^4}{6l^3} + k = \frac{2Ea^4}{l^3} + k$$

$$k_{34-1} + k_{12-2} = \frac{-3l \cdot Ea^4}{6l^3} + \frac{3l\,Ea^4}{6l^3} = 0 = k_{43-1} + k_{21-2}$$

$$k_{44-1} + k_{22-2} = \frac{2l^2 \cdot Ea^4}{6l^3} + \frac{2l^2 Ea^4}{6l^3} = \frac{2Ea^4}{3l}$$

$$m_{33-1} + m_{11-2} + M_0 = \frac{156 \cdot \varrho a^2 l}{420} + \frac{156\,\varrho a^2 l}{420} + M_0 = \frac{26\,\varrho a^2 l}{35} + M_0$$

$$m_{34-1} + m_{12-2} = \frac{-22l \cdot \varrho a^2 l}{420} + \frac{22l \cdot \varrho a^2 l}{420} = 0 = m_{43-1} + m_{21-2}$$

$$m_{44-1} + m_{22-2} = \frac{4l^2 \varrho a^2 l}{420} + \frac{4l^2 \varrho a^2 l}{420} = \frac{2\varrho a^2 l^3}{105}$$

$$\left(\begin{bmatrix} \frac{2Ea^4}{l^3}+k & 0 \\ 0 & \frac{2Ea^4}{3l} \end{bmatrix} - \omega^2 \begin{bmatrix} \frac{26\rho a^2 l}{35}+M_0 & 0 \\ 0 & \frac{2\rho a^2 l^3}{105} \end{bmatrix}\right) \cdot \begin{Bmatrix} q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\det\left(\underset{2\times 2}{[K]} - \omega^2 \underset{2\times 2}{[M]}\right) = 0$$

$$M_0 = 20\,kg = 20\,\frac{kg\,m}{s^2}\cdot\frac{s^2}{m} = 20\,\frac{Ns^2}{1000\,mm} = 0.02\,\frac{Ns^2}{mm} \qquad f_1 = 72.2\,Hz$$

$$\rho = 7800\,\frac{kg}{m^3} = 7800\,\frac{kg\,m}{s^2}\cdot\frac{s^2}{m\cdot m^3} = 7.8\cdot 10^{-9}\frac{Ns^2}{mm^4}$$

$$\left\{\left(\frac{2Ea^4}{l^3}+k\right)-\omega^2\left(\frac{26\varrho a^2 l}{35}+M_0\right)\right\}\cdot\left(\frac{2Ea^4}{3l}-\omega^2\cdot\frac{2\varrho a^2 l^3}{105}\right)=0.$$

$$\hookrightarrow \omega_1=\sqrt{\frac{\frac{2Ea^4}{l^3}+k}{\frac{26\varrho a^2 l}{35}+M_0}}=$$

$$=453.7\ 1/s$$

$$\hookrightarrow \omega_2=\sqrt{\frac{\frac{2Ea^4}{3l}}{\frac{2\varrho a^2 l^3}{105}}}=$$

$$=\frac{a}{l^2}\cdot\sqrt{\frac{35E}{\varrho}}=1198.3\ 1/s$$

$$f_1=72.2\ \mathrm{Hz}$$

$$f_2=190.71\ \mathrm{Hz}$$

EIGENVECTORS

NEW CONSTANTS :

$$\Phi(\omega_i) = \frac{2Ea^4}{l^3} + k - \omega_i^2\left(\frac{26\rho a^2 l}{35} + M_0\right)$$

$$\Psi(\omega_i) = \frac{2Ea^4}{3l} - \omega_i^2 \cdot \frac{2\rho a^2 l^3}{105}$$

$$\begin{cases} \Phi(\omega_i)\cdot q_3(\omega_i) + 0\cdot q_4(\omega_i) = 0 \\ 0\cdot q_3(\omega_i) + \Psi(\omega_i)\cdot q_4(\omega_i) = 0 \end{cases}$$

$\underline{I} + \underline{II}$:

$$\Phi(\omega_i) \cdot q_3(\omega_i) + \Psi(\omega_i) \cdot q_4(\omega_i) = 0$$

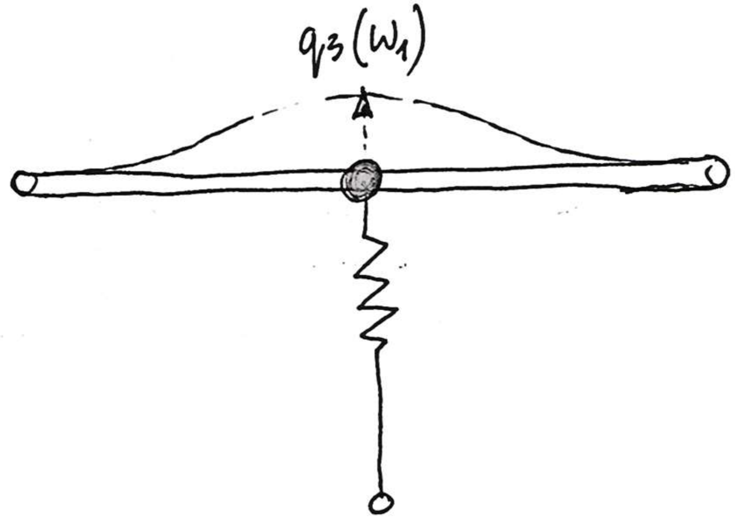
for ω_1 : $\quad \Phi(\omega_1) = 0 \quad , \quad \Psi(\omega_1) \neq 0$

$$0 \cdot q_3(\omega_1) + \Psi(\omega_1) \cdot \underbrace{q_4(\omega_1)}_{= 0} = 0$$

$q_3(\omega_1)$ → any real value

1st vibration mode

$f_1 = 72.2$ Hz



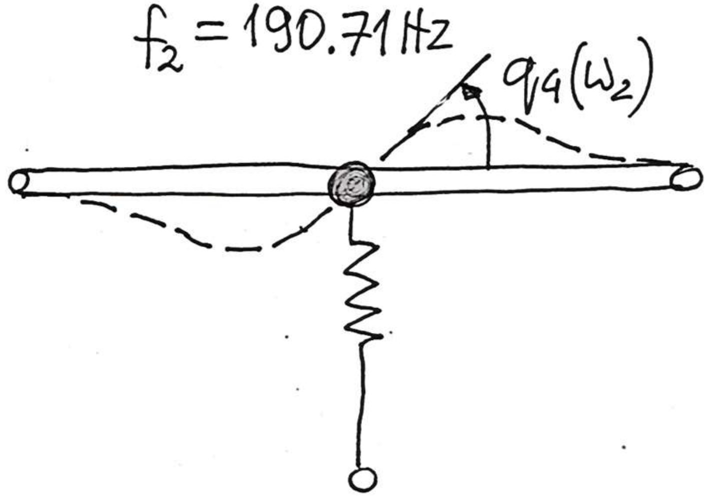
for ω_2 :

$$\Phi(\omega_2) \neq 0 \quad , \quad \Psi(\omega_2) = 0$$

$$\Phi(\omega_2) \cdot \underbrace{q_3(\omega_2)}_{=0} + 0 \cdot q_4(\omega_2) = 0$$

$q_4(\omega_2)$ → any real value

2nd vibration mode



b) for ω_2 Mass Mo only rotates:

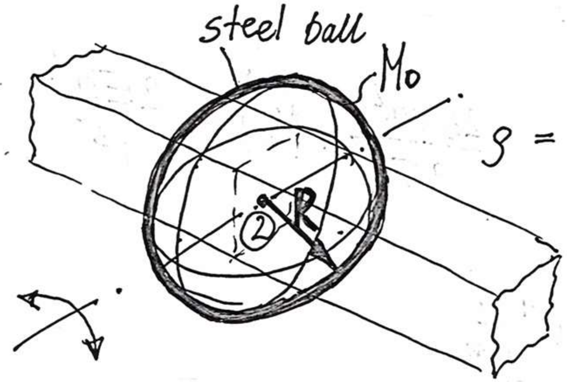
$$\tilde{\omega}_1^2 > \omega_2^2$$

$$\frac{\frac{2Ea^4}{l^3}+\tilde{k}}{\frac{26\varrho a^2 l}{35}+M_0} > \frac{35Ea^2}{\varrho l^4}$$

$$\tilde{k} > \frac{35Ea^2}{\varrho l^4}\cdot\left(\frac{26\varrho a^2 l}{35}+M_0\right)-\frac{2Ea^4}{l^3}$$

$$\tilde{k} > 4.1\cdot 10^4\,\frac{N}{mm}$$

if the moment of inertia is considered:



$$\varrho = \frac{M_0}{V} = \frac{3M_0}{4\pi R^3} \Rightarrow R = \sqrt[3]{\frac{3M_0}{4\pi R^3}} = 85\,mm$$

$$J_0 = \frac{2}{5} M_0 \cdot R^2 = \frac{2}{5} \cdot 0.02 \frac{Ns^2}{mm} \cdot 85^2 mm^2 = 57.8\,Ns^2 mm$$

$$[m']^*_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & M_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & J_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \omega_2' = \sqrt{\frac{\frac{2Ea^4}{3l}}{\frac{2\varrho a^2 l^3}{105} + J_0}} = 1074.7\,\tfrac{1}{s}$$

$$f_2' = 171\,Hz$$

$$\tilde{k}' > \frac{\frac{2Ea^4}{3l}}{\frac{2\varrho a^2 l^3}{105} + J_0} \left(\frac{26\varrho a^2 l}{35} + M_0 \right) - \frac{2Ea^4}{l^3}, \quad \tilde{k}' > 3.28 \cdot 10^4 \frac{N}{mm}$$